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On the Collision of Elastic Bodies

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X. *On the Collision of Elastic Bodies.*

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IN a paper read before the Society on June 11, Sir WILLIAM THOMSON expressed a doubt as to the general truth of the MAXWELL-BOLTZMANN doctrine concerning the distribution of energy among a great number of mutually acting bodies, and suggested that certain test cases should be investigated. The test that he proposed on that occasion was a number of hollow elastic spheres, each of mass M, and each containing a smaller elastic sphere of mass m, free to move within a larger one. This pair he calls a doublet. This case is within the general proof of the doctrine given below. It is, however, I think amenable to a simpler treatment, which has been applied to the case of elastic spheres external to one another.

1. Every doublet has a centre of inertia of the sphere M and its imprisoned m. Let V be the velocity of that centre of inertia, R the relative velocity of M and m. If V and R be given in magnitude, R given in direction, V may have any direction, and in MAXWELL'S distribution, for given direction of R, all directions of V are equally probable. Conversely, if, whatever be the values of V and R, for given direction of R all directions of V are equally probable, MAXWELL'S law prevails. Now consider a very great number of doublets, all having their relative velocity and the velocity of centre of inertia within limits R, R + dR, and V, V + dV. Consider them before and after collisions between M and m. Nothing is changed by collision except the direction of R, and that change of direction is independent of the direction of V. Therefore after collision for given direction of R all directions of V are equally probable, and therefore MAXWELL'S distribution prevails after as well as before collision, and is therefore not affected by collisions.

To proceed to more general cases.

2. The characteristic property of collisions of conventional elastic bodies is that with continuous variation of the coordinates, and without variation of the kinetic energy, the velocities at a certain instant change discontinuously. The general treatment adapted to systems of this kind is as follows:—Let there be a system defined by n coordinates $p_1 \ldots p_n$, the corresponding velocities being $\dot{p}_1 \ldots \dot{p}_n$, and the generalised components of momentum $q_1 \ldots q_n$. At a certain instant a collision, *i.e.*, discontinuity of $\dot{p}_1 \ldots \dot{p}'_n$ occurs. After the collision, let the velocities and

momenta be denoted by $\dot{p}'_1 \dots \dot{p}'_n$, $\dot{q}'_1 \dots \dot{q}'_n$. Let U be the potential. In the configuration $p_1 \dots p_n$ there are $n-1$ independent linear functions of the n forces, $-\frac{dU}{dp_1} \dots -\frac{dU}{dp_n}$, each equal to zero. We might then find n new coordinates $c_1 \dots c_n$, such that $\frac{dU}{dc_1} = 0 \dots \frac{dU}{dc_{n-1}} = 0$, and therefore if $S_1, S_2 \dots S_n$ be the components of momentum corresponding to $c_1 \dots c_n$, $\frac{dS_1}{dt} = \frac{dT}{dc_1}$ &c., and $\frac{dS_1}{dt} \dots \frac{dS_{n-1}}{dt}$ are finite or zero. In the limit when the remaining force becomes infinite and acts for an infinitely short time t, $\int_0^t \frac{dS_1}{dt} dt = 0$ or $S'_1 - S_1 = 0 \dots S'_{n-1} - S_{n-1} = 0$, and restoring the original coordinates.

$$\left.\begin{array}{l} a_1(\dot{p}_1 - \dot{p}'_1) + b_1(\dot{p}_2 - \dot{p}'_2) + \dots + k_1(\dot{p}_n - \dot{p}'_n) = 0 \\ a_2(\dot{p}_1 - \dot{p}'_1) + \dots \qquad\qquad\qquad\qquad = 0 \\ \dots \\ a_{n-} \ (\dot{p}_1 - \dot{p}'_1) + \dots \qquad\qquad\qquad\qquad = 0 \end{array}\right\} \quad . \quad . \quad . \quad (A),$$

in which the coefficients are functions of the coordinates.

And therefore $n-1$ linear functions of the $\dot{p}$'s are unaltered by the collision, namely:—

$$\left.\begin{array}{ll} a_1\dot{p}_1 + b_1\dot{p}_2 + \dots + k_1\dot{p}_n = a_1\dot{p}'_1 + b_1\dot{p}'_2 + \dots + k_1\dot{p}'_n = S_1 & \text{suppose} \\ a_2\dot{p}_1 + b_2\dot{p}_2 + \dots + k_2\dot{p}_n = a_2\dot{p}'_1 + a_2\dot{p}'_2 + \dots + k_2\dot{p}'_n = S_2 & \\ \dots & \\ a_{n-1}\dot{p}_1 + b_{n-1}\dot{p}_2 + \dots + k_{n-1}\dot{p}_n = a_{n-1}\dot{p}'_1 + b_{n-1}\dot{p}'_2 + \dots + k_{n-1}\dot{p}'_n = S_{n-1} & \end{array}\right\} . \quad (B).$$

Again, since the kinetic energy is unchanged,

$$\Sigma \dot{p}q = \Sigma \dot{p}'q',$$

and by the properties of generalised coordinates

$$\Sigma \dot{p}'q = \Sigma \dot{p}q'.$$

Therefore

$$\Sigma (q + q')(\dot{p} - \dot{p}') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C).$$

The last equation forms, with the $n-1$ equations (A), a system of n equations, all linear as regards $\dot{p}_1 - \dot{p}'_1$, &c. Since $\dot{p}_1 - \dot{p}'_1$, &c., are not all zero, we must equate the determinant of the system (A) and (C) to zero. That gives us a linear equation in $q_1 + q'_1$, $q_2 + q'_2$, &c. We can now substitute for the q's their values in terms of $\dot{p}_1 \dots \dot{p}_n$, and so obtain a linear equation

$$\alpha(\dot{p}_1 + \dot{p}'_1) + \beta(\dot{p}_2 + \dot{p}'_2), \text{ \&c.} = 0,$$

or

$$\alpha\dot{p}_1 + \beta\dot{p}_2 + \ldots = \mathrm{R} = -(\alpha\dot{p}'_1 + \beta\dot{p}'_2 \ldots),$$

where α, β, &c. are functions of the coordinates and constants.

That is, we have in all $n - 1$ linear functions of the velocities, namely, $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$, which remain unaltered by the collision, and one other linear function, R, which remains unaltered in value, but changes sign. That must be the case on every collision of elastic bodies.

3. The kinetic energy may be expressed in terms of the n variables $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$, and R, in lieu of the n velocities $\dot{p}_1 \ldots \dot{p}_n$, and since it is not altered by the collision, which changes the sign of R, leaving $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ unaltered, it must be of the form

$$2\mathrm{E} = f(\mathrm{S}_1 \ldots \mathrm{S}_{n-1}) + \lambda\mathrm{R}^2,$$

in which $f(\mathrm{S}_1 \ldots \mathrm{S}_{n-1})$ is a quadratic function of $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ with coefficients functions of the coordinates and constants of the system, and λ is a function of the coordinates and constants.

4. The system, after collision, has velocities $\dot{p}'_1$, &c., which we will call the second state. We may conceive a system with the same coordinates having velocities $-\dot{p}'_1$, $-\dot{p}'_2$, &c., and call this the second state with reversed velocities. In this state, $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ will have opposite signs to those they have in the first state, and R has the same sign as in the first state. The system retraces its course, and a collision occurs changing $-\dot{p}'_1$ into $-\dot{p}_1$, &c., leaving $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ unaltered, and changing R into $-$ R.

5. To define a collision, we may suppose that a certain function, ψ, of the coordinates and constants is prevented by the physical conditions of the system from becoming positive. When ψ becomes zero, $d\psi/dt$ from being positive becomes discontinuously negative, and a collision is said to take place. We may take ψ for one of our generalised coordinates in lieu of p_n, and $\dot{\psi}$, or $d\psi/dt$, for the corresponding component of velocity. The kinetic energy is a function of $\dot{p}_1 \ldots \dot{p}_{n-1}$, $\dot{\psi}$, and we may express it as a function of $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$, and $\dot{\psi}$, where $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ are the constants found above. Since the kinetic energy is not altered by the discontinuous change in $\dot{\psi}$, whatever the values of $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$, it must be of the form $f(\mathrm{S}_1 \ldots \mathrm{S}_{n-1}) + \frac{1}{2}\lambda\dot{\psi}^2$. That is $\dot{\psi}$ is reversed in sign, but unaltered in magnitude by the collision, and is, therefore, equal or proportional to R found above.

6. What we have proved for a system is of course true if for system we write pair of systems. For instance, let there be two sets of systems: (1) systems M defined by coordinates $p_1 \ldots p_r$, and velocities $\dot{p}_1 \ldots \dot{p}_r$, and (2) systems m defined by coordinates and velocities $p_{r+1} \ldots p_n$ and $\dot{p}_{r+1} \ldots \dot{p}_n$. If ψ is a function of $p_1 \ldots p_n$ such

that when $\psi = 0$ a collision—*i.e.*, a discontinuity of the velocities—occurs, we may treat the pair of systems M, m in all respects as a single system within the preceding investigation.

7. All those systems, or pairs of systems, for which at any instant ψ lies between zero and $-(d\psi/dt)\,\delta t$, $d\psi/dt$ being positive, will undergo collision within the time δt after that instant. We may, therefore, take $d\psi/dt$ or R as measuring the frequency of collisions for given values of $\dot{p}_1$, &c.

8. From the linear equations (B) above given we can find any of the velocities, for instance, $\dot{p}_1$, as a linear function of $S_1 \ldots S_{n-1}$ R, and $\dot{p}'_1$ will be the same function with $-$ R written for R. Hence $\dot{p}_1^2 - \dot{p}'^2_1 = 4R\Sigma\mu S$ where

$$\Sigma\mu S = \mu_1 S_1 + \mu_2 S_2 + \&c.,$$

and the μ's are functions of the coordinates and constants.

Also $(\dot{p}_1^2 - \dot{p}'^2_1)\,R = 4R^2\Sigma\mu S$.

Now, without altering E or R, or the coordinates, let us make $S_1 \ldots S_{n-1}$ pass through the whole range of values consistent with

$$2E = \lambda^2 R^2 + f(S_1 \ldots S_{n-1}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (E).$$

Also let $\phi(S_1 \ldots S_{n-1})\,dS_1 \ldots dS_{n-1}$ be the number of systems for which these variables lie between the limits

$$S_1 \text{ and } S_1 + dS,$$
$$. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad .$$
$$S_{n-1} \text{ and } S_{n-1} + dS_{n-1},$$

E and R and the coordinates being constant.

Then

$$\iint \ldots (\dot{p}_1^2 - \dot{p}'^2_1)\,R\,.\,\phi(S_1 \ldots S_{n-1})\,dS_1 \ldots dS_{n-1}$$
$$= 4R^2 \iint \ldots \phi(S_1 \ldots S_{n-1})\,\Sigma\mu S\,dS_1 \ldots dS_{n-1},$$

the integrations being over all values consistent with (E).

Now, in the MAXWELL-BOLTZMANN distribution $\phi(S_1 \ldots S_{n-1})$ is a function of the kinetic energy only, and is therefore constant throughout this integration. Therefore the right-hand member of the last equation is zero, because for any given set of values of $S_1 \ldots S_{n-1}$ satisfying (E), there is a corresponding set with the opposite signs also satisfying it. Therefore

$$\iint \ldots R\phi(S_1 \ldots S_{n-1})\,(\dot{p}_1^2 - \dot{p}'^2_1)\,dS_1 \ldots dS_{n-1} = 0,$$

and, therefore, the average value of $\dot{p}_1^2 - \dot{p}'^2_1$ for all collisions given E and R is zero. The same is true for $\dot{p}_2^2 - \dot{p}'^2_2$, &c.

And, therefore, since E and R are arbitrary, $\dot{p}_1^2$, &c., are not altered by collision at all, that is, the MAXWELL-BOLTZMANN distribution, given existing, is not altered by collisions.

The above proof also shows that $\overline{\dot{p}^2} - \overline{\dot{p}'^2_1}$ is zero without the factor R, that is, the average value for all *systems* is zero, as well as for all *collisions;* and in proving that $\overline{\dot{p}^2} = \overline{\dot{p}'^2}$, it does not matter whether we introduce the factor R or not.

9. We will give certain examples of the functions $S_1 \ldots S_{n-1}$ R.

I. Elastic spheres of masses m and M respectively. Here a colliding pair, which corresponds to a system in the general treatment, has six degrees of freedom. There should, therefore, be five linear functions of the velocity unaltered by collision. They are x, y, X, Y, the tangential components of velocity at the instant of collision, and

$$mu + \mathrm{MU} = (\mathrm{M} + m)\,\mathrm{V} = mu' + \mathrm{MU}',$$

where u, U are the normal components.

We have from the last equation

$$m\,(u - u') + \mathrm{M}\,(\mathrm{U} - \mathrm{U}') = 0,$$

and by the equation of energy

$$m\,(u^2 - u'^2) + \mathrm{M}\,(\mathrm{U}^2 - \mathrm{U}'^2) = 0,$$

whence

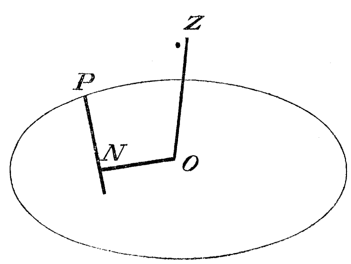
$$u + u' = \mathrm{U} + \mathrm{U}',$$

or

$$u - \mathrm{U} = -\,(u' - \mathrm{U}') = \mathrm{R}.$$

II. The system consists of a sphere of mass m colliding with a spheroid of mass M. It is assumed that the spheroid will acquire no rotation about its axis of figure, but may have rotation about any other principal axis. It has then five degrees of freedom, and the system of sphere and spheroid has eight.

We require, then, seven linear functions of the velocities to be invariable.



Let O be the centre of the spheroid, OZ in the plane of the paper, its axis of figure, P the point of collision, PN normal at P, ON perpendicular to PN, and ON $= c$, A the moment of inertia of the spheroid round an axis through O perpendicular to OZ. Then our seven constants are

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(1), (2), (3), (4). Four tangential components of velocity.

(5). The angular velocity $\dot{\theta}$ of the spheroid round an axis perpendicular to OZ in the plane of the paper.

And if u, U be normal components of velocity, and ω the angular velocity round an axis through O perpendicular to the plane of the paper, the following two, viz. :—

By conservation of momentum,

$$(6.)\quad mu + \mathrm{MU} = (\mathrm{M} + m)\,\mathrm{V} = mu' + \mathrm{MU}'.$$

By conservation of moment of momentum round the axis through O perpendicular to the plane of the paper,

$$(7.)\quad -\mathrm{M}c\mathrm{U} + \mathrm{A}\omega = \mathrm{S} = -\mathrm{M}c\mathrm{U}' + \mathrm{A}\omega'.$$

We then form the equations

$$\begin{aligned} m(u-u') + \mathrm{M}(\mathrm{U}-\mathrm{U}') &= 0,\\ -\mathrm{M}c(\mathrm{U}-\mathrm{U}') + \mathrm{A}(\omega-\omega') &= 0,\end{aligned}$$

and by conservation of energy,

$$m(u+u')(u-u') + \mathrm{M}(\mathrm{U}+\mathrm{U}')(\mathrm{U}-\mathrm{U}') + \mathrm{A}(\omega+\omega')(\omega-\omega') = 0\,;$$

and equating the determinant to zero we obtain, neglecting common factors,

$$\begin{aligned} u - \mathrm{U} - c\omega &= \rho,\\ u' - \mathrm{U}' - c\omega' &= -\rho,\end{aligned}$$

and, using V, S, and ρ for $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}$ and R in the general equations, we see that if the MAXWELL-BOLTZMANN distribution of energy exist, it is not disturbed by collisions between spheres and spheroids.

III. Professor BURNSIDE's problem (see his paper, Roy. Soc. Edinburgh, July 18, 1887). He supposes a number of similar and equal spheres, each of unit mass, but each sphere, instead of being homogeneous, has its centre of inertia at a distance c from its centre, c being supposed very small compared with the radius. The principal moments of inertia are for each sphere A, B, C, and the direction cosines of c referred to the principal axes through the centre of inertia are for each sphere the same, viz., α, β, γ. The direction cosines of the line of centres at impact referred to the principal axes are for one sphere L, M, N, and for the other l, m, n.

Further the normal velocities are U, u, and the angular velocities round the principal axes are Ω_1, Ω_2, Ω_3, ω_1, ω_2, ω_3 for the two spheres respectively.

Finally we write,

$$\begin{aligned} \mathrm{N}\beta - \mathrm{M}\gamma &= \mathrm{P} \qquad & n\beta - m\gamma &= p.\\ \mathrm{L}\gamma - \mathrm{N}\alpha &= \mathrm{Q} \qquad & l\gamma - n\alpha &= q.\\ \mathrm{M}\alpha - \mathrm{L}\beta &= \mathrm{R} \qquad & m\alpha - l\beta &= r\end{aligned}$$

The system of two spheres has twelve degrees of freedom. We require, therefore, eleven linear functions of the velocities to be invariable. They are as follows, viz., four components of velocity in the common tangent plane, x, y, X, y, and seven others, viz.,

$$\begin{aligned} u \quad &+ \mathrm{U} \quad = \mathrm{V} = u' \quad + \mathrm{U}', \\ cpu \quad &+ \mathrm{A}\omega_1 = s_1 \ = cpu' \ + \mathrm{A}\omega'_1, \\ cqu \quad &+ \mathrm{B}\omega_2 = s_2 \ = cqu' \ + \mathrm{B}\omega'_2, \\ cru \quad &+ \mathrm{C}\omega_3 = s_3 \ = cru' \ + \mathrm{C}\omega'_3, \\ c\mathrm{PU} &- \mathrm{A}\Omega_1 = \mathrm{S}_1 = c\mathrm{PU}' - \mathrm{A}\Omega'_1, \\ c\mathrm{QU} &- \mathrm{B}\Omega_2 = \mathrm{S}_2 = c\mathrm{QU}' - \mathrm{B}\Omega'_2, \\ c\mathrm{RU} &- \mathrm{C}\Omega_3 = \mathrm{S}_3 = c\mathrm{RU}' - \mathrm{C}\Omega'_3. \end{aligned}$$

As before, we form the equations,

$$\begin{aligned} u - u' + \mathrm{U} - \mathrm{U}' &= 0, \\ cp\,(u - u') + \mathrm{A}\,(\omega_1 - \omega_1) &= 0, \\ cq\,(u - u') + \mathrm{B}\,(\omega_2 - \omega'_2) &= 0, \\ cr\,(u - u') + \mathrm{C}\,(\omega_3 - \omega'_3) &= 0, \\ c\mathrm{P}\,(\mathrm{U} - \mathrm{U}') - \mathrm{A}\,(\Omega_1 - \Omega'_1) &= 0, \\ c\mathrm{Q}\,(\mathrm{U} - \mathrm{U}') - \mathrm{B}\,(\Omega_2 - \Omega'_2) &= 0, \\ c\mathrm{R}\,(\mathrm{U} - \mathrm{U}') - \mathrm{C}\,(\Omega_3 - \Omega'_3) &= 0, \end{aligned}$$

and by the conservation of energy,

$$(u + u')\,(u - u') + (\mathrm{U} + \mathrm{U}')\,(\mathrm{U} - \mathrm{U}') + \mathrm{A}\,(\omega_1 + \omega'_1)\,(\omega_1 - \omega'_1) + \&\text{c.} = 0$$

and equating the determinant to zero,

$$\begin{aligned} u - \mathrm{U} - c\,(p\omega_1 + q\omega_2 + r\omega_3 + \mathrm{P}\Omega_1 + \mathrm{Q}\Omega_2 + \mathrm{R}\Omega_3) &= \rho, \\ u' - \mathrm{U}' - c\,(p\omega'_1 + q\omega'_2 + r\omega'_3 + \mathrm{P}\Omega'_1 + \mathrm{Q}\Omega'_2 + \mathrm{R}\Omega'_3) &= -\rho \end{aligned}$$

We can now substitute V, s_1, s_2, s_3, S_1, S_2, S_3, ρ for $\mathrm{S}_1 \ldots \mathrm{S}_{n-1}\mathrm{R}$ in the general equation, and we obtain, as before, the result that the MAXWELL-BOLTZMANN distribution, given existing, is not affected by collisions.

10. Professor BURNSIDE obtains the same eight equations as above given, and I acknowledge my obligation to him, but he originally deduced the result that the energy of rotation is twice the energy of translation, instead of equal to it, as, according to the theory, it should be. He has since seen reason to change his views with regard to this problem.

The equations given by Professor BURNSIDE can easily be modified so as to meet the case of elastic bodies of any shape.

11. It would not be difficult to extend the method of (8) and show that the MAXWELL-BOLTZMANN distribution is a necessary, as well as a sufficient condition for stationary motion. But that is more completely done by following or extending BOLTZMANN'S method.

Let there be a set of systems which we will call systems M, whose co-ordinates and velocities are $p_1 \ldots p_r$ and $\dot{p}_1 \ldots \dot{p}_r$.

Let the number of such systems for which at any instant the co-ordinates lie between the limits

$$\left.\begin{array}{l} p_1 \text{ and } p_1 + dp_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_r \text{ and } p_r + dp_r \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (A),$$

and the velocities between the limits

$$\left.\begin{array}{l} \dot{p}_1 \text{ and } \dot{p}_1 + d\dot{p}_1 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dot{p}_r \text{ and } \dot{p}_r + d\dot{p}_r \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (A'),$$

be

$$F(p_1 \ldots p_r \dot{p}_1 \ldots \dot{p}_r)\, dp_1 \ldots dp_r\, d\dot{p}_1 \ldots d\dot{p}_r,$$

or, shortly,

$$F dp_1 \ldots dp_r\, d\dot{p}_1 \ldots d\dot{p}_r.$$

Let there be another set of systems, which we will call systems m, whose co-ordinates are $p_{r+1} \ldots p_n$ and velocities $\dot{p}_{r+1} \ldots \dot{p}_n$.

And let the number of systems m for which at any instant the coordinates lie between the limits

$$\left.\begin{array}{l} p_{r+1} \text{ and } p_{r+1} + dp_{r+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ p_n \text{ and } p_n + dp_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (B),$$

and the velocities between the limits

$$\left.\begin{array}{l} \dot{p}_{r+1} \text{ and } \dot{p}_{r+1} + d\dot{p}_{r+1} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ \dot{p}_n \text{ and } \dot{p}_n + d\dot{p}_n \end{array}\right\} \quad \ldots\ldots\ldots\ldots \quad (B'),$$

be

$$f(p_{r+1} \ldots p_n \dot{p}_{r+1} \ldots \dot{p}_n)\, dp_{r+1} \ldots dp_n\, d\dot{p}_{r+1} \ldots d\dot{p}_n$$

or, shortly,

$$f . dp_{r+1} \ldots dp_n \, d\dot{p}_{r+1} \ldots d\dot{p}_n.$$

The number of pairs of systems, each consisting of one system from each set, whose coordinates and velocities at any instant lie within the above limits, is

$$dp_1 \ldots d\dot{p}_r \, dp_{r+1} \ldots d\dot{p}_n \, \mathrm{F}f.$$

Now let ψ be a function of the coordinates which cannot become positive, and such that when $\psi = 0$ the velocities change discontinuously, and a collision occurs.

We may use ψ for one of our coordinates, expressing p_n in terms of $p_1 \ldots p_{n-1}$ and ψ, and in like manner $\dot{p}_n$ in terms of $\dot{p}_1 \ldots \dot{p}_{n-1}$ and $\dot{\psi}$. All those pairs for which at the given instant ψ lies between zero and $- \dot{\psi}\delta t$, $\dot{\psi}$ being positive, will undergo collision within the time δt after that instant. And so the number of such collisions in unit time is, writing R for $\dot{\psi}$,

$$dp_1 \ldots dp_{n-1} \, d\dot{p}_1 \ldots d\dot{p}_{n-1} \, d\mathrm{R} \, . \, \mathrm{F}f \, \mathrm{R}.$$

Here F is a function of $p_1 \ldots p_r$ only, but f, by virtue of the elimination of p_n and $\dot{p}_n$, is now a function $p_1 \ldots p_{n-1}$ and $\dot{p}_1 \ldots \dot{p}_{n-1}$ R.

Each of these collisions changes $\dot{p}_1$ into $\dot{p}'_1$, &c., that is, changes the system from the first into the second state. The number in unit time of collisions, which with the same coordinates change the system from the second state with reversed velocities into the first state with reversed velocities, is

$$dp_1 \ldots dp_{n-1} \, d\dot{p}'_1 \ldots d\dot{p}'_{n-1} \, d\mathrm{R} \, \mathrm{F}'f'\mathrm{R},$$

in which F′, f' are the same functions of $\dot{p}'_1$, &c., as F, f are of $\dot{p}_1$, &c.

By a known theorem

$$d\dot{p}'_1 \ldots d\dot{p}'_{n-1} = d\dot{p}_1 \ldots d\dot{p}_{n-1},$$

and in MAXWELL's distribution $\mathrm{F}f = \mathrm{F}'f'$. And so the number of reverse collisions is in that distribution equal to the number of direct collisions. And this insures the permanence of the distribution. It is assumed that there are always as many systems with any given set of velocities as with those velocities reversed. And so we speak of the second state with reversed velocities as equivalent to the second state.

12. If $\mathrm{F}'f' \neq \mathrm{F}f$, the number of reverse collisions is not equal to the number of direct collisions. And, therefore, more (or fewer) pairs of systems pass out of the first state into the second than *vice versâ*. In that case the number of systems M, whose coordinates and velocities are within the limits A, A′, is increased by collisions

with systems m, whose coordinates and velocities lie within the limits B, B′, by the quantity

$$dp_1 \ldots dp_{n-1}\, d\dot{p}_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R}\, (\mathrm{F}'f' - \mathrm{F}f)\, \mathrm{R} \text{ per unit of time,}$$

and is increased per unit of time by collisions with systems m for all values of $p_{r+1} \ldots \dot{p}_n$ by the quantity

$$dp_1 \ldots dp_r\, d\dot{p}_1 \ldots d\dot{p}_r \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\, \mathrm{R}\, dp_{r+1} \ldots d\dot{p}_{n-1}\, d\mathrm{R},$$

all values of p_{r+1}, &c., being included in the integration.

We will now assume (see p. 418, *post*) that the velocities of M and m are not on average altered except by collision between M and m. And so the above-mentioned increments are the only increments by which the class of M systems within the limits A, A′ is affected.

In that case

$$\frac{d\mathrm{F}}{dt}\, dp_1 \ldots d\dot{p}_r = dp_1 \ldots d\dot{p}_r \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\, \mathrm{R}\, dp_{r+1} \ldots d\dot{p}_{n-1}\, d\mathrm{R},$$

and, therefore,

$$\iint \ldots \frac{d\mathrm{F}}{dt} \log \mathrm{F}\, dp_1 \ldots d\dot{p}_r \quad (\text{over all values of } p_1 \ldots \dot{p}_r)$$

$$= \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\, \mathrm{R} \log \mathrm{F}\, dp_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R}.$$

By symmetry, as the right-hand member includes all possible collision between M and m,

$$\iint \ldots \frac{df}{dt} \log f\, dp_{r+1} \ldots d\dot{p}_n$$

$$= \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\, \mathrm{R} \log f\, dp_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R},$$

and, therefore,

$$\iint \ldots \frac{d\mathrm{F}}{dt} \log \mathrm{F}\, dp_1 \ldots d\dot{p}_r + \iint \ldots \frac{df}{dt} \log f\, dp_{r+1} \ldots d\dot{p}_n$$

$$= \iint \ldots (\mathrm{F}'f' - \mathrm{F}f)\, \mathrm{R} \log (\mathrm{F}f)\, dp_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R}.$$

By symmetry, as we may interchange the accents,

$$\iint \ldots \frac{d\mathrm{F}}{dt} \log \mathrm{F}\, dp_1 \ldots d\dot{p}_r + \iint \ldots \frac{df}{dt} \log f\, dp_{r+1} \ldots d\dot{p}_n$$

$$= \iiint \ldots (\mathrm{F}f - \mathrm{F}'f')\, \mathrm{R} \log (\mathrm{F}'f')\, dp_1 \ldots d\dot{p}_{n-1}\, d\mathrm{R},$$

and, therefore,

$$= \tfrac{1}{2} \iiint \ldots (F'f' - Ff)\, R \log \frac{Ff}{F'f'}\, dp_1 \ldots d\dot{p}_{n-1}\, dR.$$

Now let

$$H = \iiint \ldots F(\log F - 1)\, dp_1 \ldots d\dot{p}_r + \iiint \ldots f(\log f - 1)\, dp_{r+1} \ldots d\dot{p}_n,$$

and, therefore,

$$\frac{dH}{dt} = \iiint \ldots \frac{dF}{dt} \log F\, dp_1 \ldots d\dot{p}_r + \iiint \ldots \frac{df}{dt} \log f\, dp_{r+1} \ldots d\dot{p}_n$$

$$= \tfrac{1}{2} \iiint \ldots (F'f' - Ff)\, R \log \frac{Ff}{F'f'}\, dp_1 \ldots d\dot{p}_{n-1}\, dR,$$

as we have seen.

Now, this expression is necessarily negative, unless $F'f' = Ff$ whenever the pair of systems having coordinates and velocities $p_1 \ldots \dot{p}_n$ can pass by collision, and, therefore, with unchanged kinetic energy, into the state in which they are $p_1 \ldots \dot{p}'_n$, that is, unless the MAXWELL-BOLTZMANN distribution exist, and is then zero. H therefore tends to a minimum which it reaches when $Ff = F'f'$.

We will now make

$$H = H_1 + K$$

where H_1 is the minimum value assumed by H when $F'f' = Ff$. Then $dH/dt = 0$, and $dK/dt = dH/dt$, and is always negative. We may define the function K to be *the disturbance*, and $(1/K)\ (dK/dt)$ to be the *rate of subsidence* of the disturbance by collision. In certain cases, we can calculate the rate.

13. We have assumed that f varies only as the result of collisions. That is, if $\partial f/\partial t$ denote the time variation of f due to causes other than collisions, and $\partial H/\partial t$ be formed from $\partial f/\partial t$ as dH/dt from df/dt, then $\partial H/\partial t = 0$, on average. It is worth while to consider on what condition this may be safely assumed.

Let

$$\frac{\partial H}{\partial t} = \iiint \ldots \frac{\partial f}{\partial t} \log f\, dp_1 \ldots d\dot{p}_n.$$

As we are dealing with rigid elastic bodies under the action of no forces, we may treat f as a function of three translation velocities, and three angular velocities, w_1, w_2, w_3, about three principal axes of the solids. Let A, B, C, be the principal moments of inertia. Evidently, there being no forces, the translation velocities cannot vary except as the result of collisions. But for each solid, w_1, w_2, w_3 may vary, the law of their variations being EULER's equations. We may, therefore, in calculating $\partial H/\partial t$ treat f as a function of w_1, w_2, w_3 only. Then

$$\frac{\partial f}{\partial t} = \frac{df}{dw_1}\frac{dw_1}{dt} + \frac{df}{dw_2}\frac{dw_2}{dt} + \frac{df}{dw_3}\frac{dw_3}{dt}$$

$$= \frac{B-C}{A}\frac{df}{dw_1} w_2 w_3 + \frac{C-A}{B}\frac{df}{dw_2} w_1 w_3 + \frac{A-B}{C}\frac{df}{dw_3} w_1 w_2,$$

and

$$\frac{\partial H}{\partial t} = \frac{B-C}{A}\iiint \frac{df}{dw_1}\log f \,.\, w_2 w_3 \,.\, dw_1\, dw_2\, dw_3,$$

with two other corresponding terms, the limits being in each $\pm\infty$. Now, with these limits,

$$\iiint \frac{df}{dw_1}\log f \,.\, w_2 w_3 \,.\, dw_1\, dw_2\, dw_3$$

$$= \iint dw_2\, dw_3\, w_2 w_3 \{f\log f_{w_1=\infty} - f\log f_{w_1=-\infty}\}.$$

Now, we may assume $f = 0$ and $f\log f = 0$, when any one of the three variables is infinite, whether positive or negative. And this assumption is sufficient to justify the statement $\partial H/\partial t = 0$.

14. It is possible to calculate, in a simple case, the rate at which a disturbance subsides by collisions. For example, two sets of elastic spheres, N of mass M, and n of mass m, in unit of volume. In the normal state, the number in unit of volume, whose velocities are represented by lines drawn from an origin to points within an element of volume $U^2 \sin\theta\, d\theta\, d\phi\, dU$ is for the M spheres

$$N\left(\frac{hM}{\pi}\right)^{\frac{3}{2}} \epsilon^{-hMU^2}\, U^2 \sin\theta\, d\theta\, d\phi\, dU,$$

where U, θ, ϕ are usual spherical coordinates; or, let us say,

$$F(U) = N\left(\frac{hM}{\pi}\right)^{\frac{3}{2}} \epsilon^{-hMU^2}.$$

Similarly, for the m spheres,

$$f(u) = n\left(\frac{hm}{\pi}\right)^{\frac{3}{2}} \epsilon^{-hmu^2}$$

expresses the law of distribution of velocities in the undisturbed state. We will write F and f for these expressions.

We will now suppose there is a small disturbance consisting in h having different values for the two sets. Let h be written $h(1 + D)$ for the M spheres, and $h(1 + d)$ for the m spheres. We shall neglect third and higher powers of D, d. Then F becomes in the disturbed state

$$N\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}(1 + D)^{\frac{3}{2}} \epsilon^{-h(1+D)MU^2},$$

that is,

$$N\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}(1+D)^{\frac{3}{2}}\epsilon^{-hMU^2}\{1-hDMU^2+\text{terms in }D^2\}.$$

(It will appear that terms in D^2, &c., are not required.)

Similarly f becomes

$$n\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}(1+d)^{\frac{3}{2}}\epsilon^{-hmu^2}(1-dhmu^2+d^2\text{ \&c.}).$$

We will further suppose that the disturbance is introduced without changing the total energy. That gives the relation

$$N\,.\,\frac{3}{2h(1+D)}+n\frac{3}{2h(1+D)}=\frac{3}{2h}(N+n),$$

or

$$\frac{N}{1+D}+\frac{n}{1+d}=N+n.$$

15. The disturbance will subside by collisions between M and m. And we will treat of the case in which it subsides in such manner that the above values of F and f apply at every instant with the values that D and d at that instant have. Such a mode of subsidence is possible, at all events if our equations lead (as they do) to a relation of the form $(1/K)\,(dK/dt)=$ constant.

Let us then form the function

$$H=\int_0^\infty\int_0^\pi\int_0^{2\pi}.\,F(\log F-1)\,U^2\sin\alpha\,d\alpha\,d\beta\,dU,$$

where U, α, and β are usual spherical coordinates,

$$+\int_0^\infty\int_0^\pi\int_0^{2\pi}f(\log f-1)\,u^2\sin\alpha\,d\alpha\,d\beta\,du,$$

where F and f have the values above given.

That is

$$H=\tfrac{3}{2}\log(1+D)\iiint F.U^2\sin\alpha\,d\alpha\,d\beta\,dU$$

$$+\tfrac{3}{2}\log(1+d)\iiint f.u^2\sin\alpha\,d\alpha\,d\beta\,du$$

$$+\text{ terms independent of D and }d,$$

or

$$H=N\tfrac{3}{2}\log(1+D)+n\tfrac{3}{2}\log(1+d),$$
$$+\text{ terms independent of D and }d,$$

because

$$\iiint FU^2\sin\alpha\,d\alpha\,d\beta\,dU=N$$

$$\iiint fu^2\sin\alpha\,d\alpha\,d\beta\,du=n.$$

The terms independent of D and d are equivalent to H_1, the minimum value of H when there is no disturbance.

And so

$$K = H - H_1 = \tfrac{3}{2} N \log (1 + D) + \tfrac{3}{2} n \log (1 + d).$$

That is

$$K = \tfrac{3}{2}\, n \left(d - \frac{d^2}{2}\right) + \tfrac{3}{2}\, N \left(D - \frac{D^2}{2}\right).$$

Now since

$$\frac{N}{1 + D} + \frac{n}{1 + d} = N + n,$$

$$D = -\frac{nd}{N + (N + n)\, d},$$

and

$$K = \tfrac{3}{2}\, n \left(d - \frac{d^2}{2}\right) - \tfrac{3}{2}\, N \,.\, \frac{nd}{N + (N + n)\, d} - \tfrac{3}{4} \,.\, \frac{n^2 d^2}{\{N + (N + n)\, d\}^2}$$

$$= \tfrac{3}{2}\, n \left(d - \frac{d^2}{2}\right) - \tfrac{3}{2}\, nd \,.\, \left(1 - \frac{N + n}{N}\, d\right) - \tfrac{3}{4}\, \frac{n^2 d^2}{\{N + (N + n)\, d\}^2}$$

$$= \tfrac{3}{2}\, nd^2 - \tfrac{3}{4}\, nd^2 + \tfrac{3}{2}\, \frac{n^2}{N}\, d^2 - \tfrac{3}{4}\, \frac{n^2}{N}\, d^2$$

$$= \tfrac{3}{4}\, \frac{n}{N}\, (N + n)\, d^2.$$

In order to find dK/dt we will transform our coordinates thus: Let V denote the velocity of the centre of inertia of a pair of spheres M and m, ρ their relative velocity, θ the angle between V and ρ. Then

$$U^2 = V^2 + \left(\frac{m}{M + m}\, \rho\right)^2 + \frac{2m}{M + m}\, V\rho \cos \theta$$

$$u^2 = V^2 + \left(\frac{M}{M + m}\, \rho\right)^2 - \frac{2M}{M + m}\, V\rho \cos \theta$$

and

$$MU^2 + mu^2 = (M + m)\, V^2 + \left(\frac{Mm}{M + m}\right) \rho^2$$

and so

$$Ff = Nn \left(\frac{hM}{\pi}\right)^{\frac{3}{2}} \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} (1 + D)^{\frac{3}{2}} (1 + d)^{\frac{3}{2}}\, \epsilon^{-h \left\{(M + m)\, V^2 + \frac{Mm}{M + m} \rho^2\right\}} (1 - DhMU^2 - dhmu^2)$$

$$F'f' = Nn \left(\frac{hM}{\pi}\right)^{\frac{3}{2}} \left(\frac{hm}{\pi}\right)^{\frac{3}{2}} (1 + D)^{\frac{3}{2}} (1 + d)^{\frac{3}{2}}\, \epsilon^{-h \left\{(M + m)\, V^2 + \frac{Mm}{M + m} \rho^2\right\}} (1 - DhMU'^2 - dhmu'^2)$$

where U′, u' are the values of U, u after collision.

Therefore

$$Ff - F'f' = Nn\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}(1 + D)^{\frac{3}{2}}(1 + d)^{\frac{3}{2}}\epsilon^{-h\left\{(M+m)V^2 + \frac{Mm}{M+m}\rho^2\right\}}$$
$$\{DhM(U'^2 - U^2) + dhm(u'^2 - u^2)\}$$

Now, if θ' be what θ becomes after collision,

$$U'^2 - U^2 = 2V\rho \cdot \frac{m}{M + m}(\cos\theta' - \cos\theta)$$

$$u'^2 - u^2 = -2V\rho\frac{M}{M + m}(\cos\theta' - \cos\theta)$$

Also by the relation

$$\frac{N}{1 + D} + \frac{n}{1 + d} = N + n$$

$$D = -\frac{nd}{N + (N + n)d},$$

and making these substitutions

$$F'f' - Ff = Nn\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}(1 + D)^{\frac{3}{2}}(1 + d)^{\frac{3}{2}}\epsilon^{-h\left\{(M+m)V^2 + \frac{Mm}{M+m}\rho^2\right\}}$$
$$\left(h\frac{Mm}{M + m}2 \cdot V\rho \cdot \frac{N + n}{N + (N + n)d}(\cos\theta' - \cos\theta)d\right).$$

Also

$$\log\frac{Ff}{F'f'} = h\frac{Mm}{M + m}2V\rho\frac{N + n}{N + (N + n)d}(\cos\theta - \cos\theta')d.$$

In forming $(F'f' - Ff)\log(Ff/F'f')$ we see that the last factor is squared, and so the product contains the factor d^2. We may, therefore, now write 1 for $(1 + D)^{\frac{3}{2}}$ and $(1 + d)^{\frac{3}{2}}$, and also write $(N + n)/N$ for $(N + n)/\{N + (N + n)d\}$, otherwise we should have terms in d^3.

Therefore

$$(F'f' - Ff)\log\frac{Ff}{F'f'} = -Nn\left(\frac{hM}{\pi}\right)^{\frac{3}{2}}\left(\frac{hm}{\pi}\right)^{\frac{3}{2}}\epsilon^{-h\left\{(M+m)V^2 + \frac{Mm}{M+m}\rho^2\right\}}$$
$$h^2 d^2\left(\frac{Mm}{M + m}\right)^2 4V^2\rho^2\frac{(N + n)^2}{N^2}(\cos\theta' - \cos\theta)^2.$$

Again, dK/dt contains the factor $\dot{\psi}$ denoting the frequency of collision. Now the number of collisions given V and ρ in unit of volume and time is proportional to $\pi s^2\rho$, where s is the sum of the radii of M and m. Also all directions of ρ before collision are equally probable, and given the direction before collision, all directions after collision are equally probable. Therefore, given V and ρ, the number in unit of volume and time of colliding pairs for which the angle between V and ρ before collision is between θ and $\theta + d\theta$ is $\frac{1}{2}\pi s^2 \rho \sin\theta \, d\theta$, and the number for which after collision it lies between θ' and $\theta' + d\theta''$ is $\frac{1}{2}\pi s^2\rho \sin\theta' \, d\theta'$. Hence we have to multiply $(\cos\theta' - \cos\theta)^2$ by $\frac{1}{4}\pi s^2 \rho \sin \theta \, d\theta \sin\theta' \, d\theta'$, and integrate in each case from π to 0. The result is $\frac{2}{3}\pi s^2 \rho$.

And so we get

$$\frac{d\mathrm{K}}{dt} = -\tfrac{2}{3}\,\mathrm{N}n\frac{(\mathrm{N}+n)^2}{\mathrm{N}^2}\;h^2 d^2 \left(\frac{\mathrm{M}m}{\mathrm{M}+m}\right)^2 \left(\frac{h\mathrm{M}}{\pi}\right)^{\frac{3}{2}} \left(\frac{hm}{\pi}\right)^{\frac{3}{2}}$$

$$\times \int_0^\infty\!\int_0^\pi\!\int_0^{2\pi} \mathrm{V}^2\,d\mathrm{V}\sin\alpha\,d\alpha\,d\beta \int_0^\infty d\rho\; 4\pi\rho^2\,\epsilon^{-h\left\{(\mathrm{M}+m)\mathrm{V}^2 + \frac{\mathrm{M}m}{\mathrm{M}+m}\rho^2\right\}}\;4\mathrm{V}^2\rho^3\,.\,\pi s^2,$$

where V, α, β are usual spherical coordinates

$$= -\frac{4}{\sqrt{\pi}}\frac{n}{\mathrm{N}}(\mathrm{N}+n^2)\frac{\sqrt{(\mathrm{M}m)}}{(\mathrm{M}+m)^{\frac{3}{2}}}\frac{\pi s^2}{\sqrt{h}}d^2.$$

Also, as we have seen

$$\mathrm{K} = \tfrac{3}{4}\frac{n}{\mathrm{N}}(\mathrm{N}+n)\,d^2,$$

and therefore

$$\frac{1}{\mathrm{K}}\frac{d\mathrm{K}}{dt} = -\frac{16}{3\sqrt{\pi}}(\mathrm{N}+n)\frac{\sqrt{(\mathrm{M}m)}}{(\mathrm{M}+m)^{\frac{3}{2}}}\frac{\pi s^2}{\sqrt{h}}$$

$$= -\mathrm{C} \text{ suppose.}$$

And if K_0, D_0, d_0 be initial values,

$$\mathrm{K} = \mathrm{K}_0\epsilon^{-ct}, \qquad \mathrm{D} = \mathrm{D}_0\epsilon^{-\frac{1}{2}ct}, \qquad d = d_0\epsilon^{-\frac{1}{2}ct},$$

the rate of subsidence is directly proportional to the density and to the square root of the absolute temperature.*

* Since writing the above I find that this result has already been obtained for the case of elastic spheres by Professor TAIT, by an independent method.